Integral Representation of Zero-Memory Nonlinear Functions

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Integral representation of zero-memory nonlinear functions offers promise as an analytical method for nonlinear control systems study. A review of work performed at Bell Laboratories and elsewhere on the use of these representations is presented, with particular emphasis on nonlinearities often encountered in feedback control systems. In general, the integral representations are useful only insofar as the resulting expression can be readily evaluated. The use of Bennett functions systematized the formulation of these integrals. The numerical results of a large class of the integrals can then be given by the tabulated Bennett functions. A comprehensive bibliography is appended.

I. INTRODUCTION

Integral representation of zero-memory nonlinear functions has been extensively used by Bennett, Rice and others (see References) in the solving of problems such as the finding of modulation products when one or more sinusoids appear at the input, and the finding of the output autocovariance function when sine wave and random noise are applied. In relation to the necessary calculations which occur in the use of these integral representations, a class of functions known as Bennett functions, after W. R. Bennett, has been defined. A selected representation of these functions has been tabulated and plotted.

While the original studies were carried out in relation to problems encountered in communications, the methods and the results can certainly be applied to advantage in control problems. Some work in this regard has been done by J. C. Lozier in unpublished notes on the analysis of the oscillating control servomechanism. On the whole, however, it appears that these approaches are not known to investigators in the controls field. The present paper represents an attempt to summarize in a unified manner the work that has been done and to indicate the

scope of applications and limitations of the integral representations, particularly with respect to controls usage.

II. INTEGRAL REPRESENTATION ARISING FROM FOURIER TRANSFORMS

It is known that the function

$$f_1(u) = \frac{u}{2} + \frac{u}{\pi} \int_0^\infty \frac{\sin u\lambda}{\lambda}\, d\lambda \tag{1}$$

is discontinuous in its first derivative with respect to u, its value as a function of u being:

$$\begin{aligned} f_1(u) &= u \qquad u > 0 \\ &= 0 \qquad u < 0. \end{aligned} \tag{2}$$

The plot of $f_1(u)$ vs u is in the form of an ideal half-wave rectifier.

Using $f_1(u)$ as a basic unit, other discontinuous functions can be generated. For example

$$\begin{aligned} f_2(u) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin u\lambda}{\lambda}\, d\lambda &= 1 \qquad u > 0 \\ &= 0 \qquad u < 0 \end{aligned} \tag{3}$$

and is in the form of an off-on relay as a function of u.

From (3) the bang-bang type of relay is readily created as:

$$\begin{aligned} f_3(u) = \frac{2A}{\pi} \int_0^\infty \frac{\sin u\lambda}{\lambda} &= A \qquad u > 0 \\ &= -A \quad u < 0. \end{aligned} \tag{4}$$

A relay with dead zone is

$$\begin{aligned} f_4(u) &= \frac{A}{\pi} \int_0^\infty \frac{\sin (u - c)\lambda + \sin (u + c)\lambda}{\lambda}\, d\lambda \\ &= \frac{2A}{\pi} \int_0^\infty \frac{\sin u\lambda \cos c\lambda}{\lambda}\, d\lambda \begin{array}{lr} = A & u > c \\ = 0 & -c < u < c \\ = -A & u < -c. \end{array} \end{aligned} \tag{5}$$

A limiter (linear characteristic with saturation) is

$$f_5(u) = \frac{2}{\pi} \int_0^\infty \frac{\sin \mu\lambda \sin A\lambda}{\lambda^2}\, d\lambda \begin{array}{lr} = -A & u < -A \\ = u & -A < u < A \\ = A & u > A \end{array} \tag{6a}$$

$$= \frac{2u}{\pi} \int_0^\infty \frac{\cos u\lambda \sin A\lambda}{\lambda}\, d\lambda + \frac{2A}{\pi} \int_0^\infty \frac{\sin u\lambda \cos A\lambda}{\lambda}\, d\lambda. \tag{6b}$$

Equation (6b) is readily obtained by manipulating two functions of the form of (1). That (6a) is equivalent to (6b) is seen by integrating (6a) once by parts. Other discontinuous functions can be generated from the above five functions by appropriate shifting (bias) of each individual characteristic, or by combining several characteristics. In fact, simply multiplying by an appropriate $g(u)$ can create quite general discontinuous characteristics.

It is noted that u may be viewed as the input to the nonlinear element, and $f(u)$ then gives the response to this input. If $u(t)$ is a function of time, for each $u(t_1)$ the function $f[u(t_1)]$ yields the instantaneous value of the output (i.e., $f(u)$ is a functional of u). While $f(u)$ is no more convenient for use in the evaluation of the output as a function of time than equations of the form (2), giving the discontinuous function as a set of equations, it is very useful for the purpose of spectral analysis since $f[u(t)]$ is in a compact form suitable for Fourier series expansion.

As an example, we seek to find the output spectral component for the relay with dead zone, when input is in the form $u = P \cos x$.

Using (5), the output Fourier coefficients are found by:

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left\{ \frac{2A}{\pi} \int_0^\infty \frac{\sin (\lambda P \cos x) \cos c\lambda}{\lambda} \, d\lambda \right\} \cos nx \, dx \\
&= \frac{2A}{\pi^2} \int_0^\infty \frac{d\lambda}{\lambda} \cos c\lambda \int_{-\pi}^{\pi} \sin (\lambda P \cos x) \cos nx \, dx \\
&= \begin{cases} 0 & n \text{ even} \\ \dfrac{A}{4\pi} (-1)^{(n-1)/2} \displaystyle\int_0^\infty \frac{J_n(\lambda P) \cos c\lambda}{\lambda} \, d\lambda & n \text{ odd} \end{cases}
\end{aligned}
$$

where $J_n(z)$ is Bessel function of the first kind of order n.

Since it is known that[1]

$$
\int_0^\infty \frac{J_n(at) \cos bt}{t} \, dt = \begin{cases} \dfrac{1}{n} \cos \left\{ n \sin^{-1} \left(\dfrac{b}{a} \right) \right\} & b < a \\[2ex] \dfrac{a^n \cos \dfrac{n\pi}{2}}{n\{b + \sqrt{b^2 - a^2}\}^n} & b > a, \end{cases} \tag{7}
$$

then, for n odd:

$$
\begin{aligned}
a_n &= \frac{A}{4\pi} (-1)^{(n-1)/2} \left\{ \frac{1}{n} \cos \left[n \sin^{-1} \left(\frac{c}{P} \right) \right] \right\} & c < P \\
&= 0 & P < c.
\end{aligned}
$$

Certain of the nonlinear characteristics expressed in the integral form are also amenable to a double frequency type of analysis in which the input is of the form

$$u(t) = P \cos (\omega_1 t + \theta_1) + Q \cos (\omega_2 t + \theta_2).$$

Bennett[2] in particular has contributed extensively to double frequency studies.

In control systems analysis, a double frequency study becomes necessary in (a) the oscillating servomechanisms[3] and (b) the dual input describing function approach to closed loop servos.[4,5,6] In what follows, the fundamental components (i.e., components in ω_1 and ω_2 of the output) from a bang-bang type of relay are found. The approach follows closely that of Lozier.

The input u is a function of two frequencies ω_1 and ω_2 ; this is brought to light by setting $x = \omega_1 t + \theta_1$, $y = \omega_2 t + \theta_2$, and letting $Q/P = k$. Thus

$$u(x, y) = P(\cos x + k \cos y). \tag{8}$$

In passing through a bang-bang type relay, it is recognized that the amplitude P in (8) does not influence the output; thus without loss of generality it may be set to unity.

The output $f(u)$, written as $f(x,y)$ is:

$$f(x,y) = +A \qquad \cos x + k \cos y > 0$$

$$f(x,y) = -A \qquad \cos x + k \cos y < 0$$

which may be expressed as a double Fourier series[2,7] as:

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [A_{\pm mn} \cos (mx \pm ny) + B_{\pm mn} \sin (mx \pm ny)] \tag{9}$$

where

$$A_{\pm mn} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos (mx \pm ny) \, dy \, dx \tag{9a}$$

$$B_{\pm mn} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin (mx \pm ny) \, dy \, dx \tag{9b}$$

$$A_{00} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \, dy \, dx, \qquad B_{00} = 0. \tag{9c}$$

From symmetry of the bang-bang relay, $B_{\pm mn} \equiv 0$ for all m and n; moreover the integral representation of (4) can here be used, thus:

$$A_{\pm mn} = \frac{A}{\pi^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\int_0^\infty \frac{d\lambda \sin (\cos x + k \cos y) \lambda}{\lambda} \right] \cdot \cos (mx \pm ny)\, dy\, dx.$$

The interchange of integration can be carried out here in view of the finite limits of the outer integral and the bounded nature of the inner integral, whence:

$$A_{\pm mn} = \frac{A}{\pi^3} \int_0^\infty \frac{d\lambda}{\lambda} \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\ \{\sin [(\cos x + k \cos y)\lambda] \cos (mx \pm ny)\}.$$

Upon expanding, collecting nonzero terms, and integrating, the result is (as $A_{+mn} = A_{-mn}$ for all m, n, the $\pm$ signs are henceforth dropped.):

$$\begin{aligned} A_{mn} &= \frac{4A}{\pi} (-1)^{(m+n-1)/2} \int_0^\infty \frac{J_n(k\lambda) J_m(\lambda)}{\lambda}\, d\lambda, \qquad && m + n \text{ odd} \\ &= 0 && \text{otherwise} \end{aligned} \tag{10}$$

where use has been made of the following definite integrals:[8,1]

$$\frac{2}{\pi} \int_0^{\pi/2} \cos (z \sin \varphi) \cos 2n\varphi d\varphi = (-1)^n \frac{2}{\pi} \int_0^{\pi/2} \cos (z \cos \varphi) \cos (2n\varphi)\, d\varphi = J_{2n}(z) \tag{11a}$$

$$\frac{2}{\pi} \int_0^{\pi/2} \sin (z \sin \varphi) \sin (2n + 1)\varphi d\varphi = (-1)^n \frac{2}{\pi} \int_0^{\pi/2} \sin (z \cos \varphi) \cos (2n + 1)\varphi d\varphi = J_{2n+1}(z). \tag{11b}$$

The integral in (10) may be evaluated by means of formulas attributed to Sonine and Schafheitlin[9] (also known as the Weber-Schafheitlin integrals[8]), the result being expressed in the form of hypergeometric functions $F(\alpha,\beta,c,x)$:

$$\int_0^\infty \frac{J_n(a\lambda) J_m(b\lambda)}{\lambda^r}\, d\lambda = \frac{a^n \Gamma\left(\dfrac{n + m - r + 1}{2}\right)}{2^r b^{n-r+1}\, \Gamma\left(\dfrac{-n + m + r + 1}{2}\right) \Gamma(n + 1)} \cdot F\left(\frac{n + m - r + 1}{2}, \frac{n - m - r + 1}{2}, n + 1, \left(\frac{a}{b}\right)^2\right) \tag{12a}$$

if $n + m - r + 1 > 0$, $r > -1$, and $0 < a < b$,

$$\int_0^\infty \frac{J_n(a\lambda)J_m(a\lambda)}{\lambda^r}\,d\lambda$$

$$= \frac{\left(\frac{a}{2}\right)^{r-1}\Gamma(r)\,\Gamma\left(\frac{n+m-r+1}{2}\right)}{2\Gamma\left(\frac{-n+m+r+1}{2}\right)\Gamma\left(\frac{n+m+r+1}{2}\right)\Gamma\left(\frac{n-m+r+1}{2}\right)} \tag{12b}$$

if $n + m + 1 > 0$, $r > 0$, and a real; and

$$\int_0^\infty \frac{J_n(a\lambda)J_m(b\lambda)}{\lambda^r}\,d\lambda = \frac{b^m\Gamma\left(\frac{n+m-r+1}{2}\right)}{2^r a^{m-r+1}\Gamma\left(\frac{n-m+r+1}{2}\right)\Gamma(m+1)} \cdot F\left(\frac{n+m-r+1}{2}, \frac{-n+m-r+1}{2}, m+1, \frac{b^2}{a^2}\right) \tag{12c}$$

if $(n + m - r + 1) > 0$, $r > -1$, and $0 < b < a$.

Accordingly,

$$A_{mn} = \frac{2A}{\pi}(-1)^{(n+m-1)/2}\frac{k^n\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{2-n+m}{2}\right)\Gamma(n+1)} \cdot F\left(\frac{n+m}{2}, \frac{n-m}{2}, n+1, k^2\right) \qquad \text{for } k < 1 \tag{13a}$$

$$= \frac{2A}{\pi}(-1)^{(n+m-1)/2} \cdot \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{2-n+m}{2}\right)\Gamma\left(\frac{2+n+m}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)} \qquad \text{for } k = 1 \tag{13b}$$

$$= \frac{2A}{\pi}(-1)^{(n+n-1)/2}\frac{\Gamma\left(\frac{n+m}{2}\right)}{k^m\Gamma\left(\frac{n-m+2}{2}\right)\Gamma(m+1)} \cdot F\left(\frac{n+m}{2}, \frac{m-n}{2}, m+1, \left(\frac{1}{k}\right)^2\right) \qquad \text{for } k > 1. \tag{13c}$$

The three cases of (13) are essentially equivalent if one recognizes that

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

and that, for $k > 1$, the situation is identical to that of inverting the role of n and m, and defining a new quantity $k' = 1/k$.

The output fundamental components are A_{10} and A_{01} ; from (13) one has:

$$A_{10} = \frac{2A}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(1)} F\left(\frac{1}{2}, -\frac{1}{2}, 1, k^2\right) \tag{14a}$$

$$= \frac{4A}{\pi}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 \cdots\right)$$

$$A_{01} = \frac{2A}{\pi} \frac{k\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2)} F\left(\frac{1}{2}, \frac{1}{2}, 2, k^2\right) \tag{14b}$$

$$= \frac{2Ak}{\pi}\left(1 + \frac{1}{8}k^2 + \frac{3}{64}k^4 \cdots\right).^\dagger$$

Considered together with the input (8), this yields Lozier's oft-quoted result,[3,10] that the equivalent "gain" of the relay, for small values of k, is 6 db higher for the "carrier" than for the "signal."

It is not difficult to see that the Weber-Schafheitlin integrals also occur for two frequency inputs applied to the characteristics (1) and (3), but that integrals of the form

$$\int_0^\infty \frac{J_n(a\lambda)J_m(b\lambda)}{\lambda^r} \begin{Bmatrix}\sin A\lambda \\ \cos A\lambda\end{Bmatrix} d\lambda \tag{15}$$

occur for characteristics of (5) and (6). Moreover, inputs with more than two frequency components will result in generalized Weber-Schafheitlin integrals of the form

$$\int_0^\infty \prod_{i=1}^{N} J_i(a_i\lambda)\lambda^{-r}\, d\lambda \tag{16}$$

† Equations (14a) and (14b) can also be expressed in terms of the complete elliptic integrals for which tables are available. See Refs. 3, 21, and 22.

for characteristics (1), (3), and (4), and generalized integrals of the form

$$\int_0^\infty \prod_{i=1}^N J_i(a_i\lambda)\lambda^{-r} \begin{Bmatrix} \sin A\lambda \\ \cos A\lambda \end{Bmatrix} d\lambda \tag{17}$$

result for characteristics (5) and (6).

Unfortunately no general solutions have been found to represent (15), (16) or (17) in known functions. In such cases, numerical solutions can be used. Numerical solutions in terms of Bennett functions and their tabulation are described in Section IV of this paper.

III. INTEGRAL REPRESENTATION ARISING FROM LAPLACE TRANSFORM[11,12]

The integral representation of Section II is closely related to the Fourier transform. An alternate approach using the Laplace transform is more convenient in some cases and has been extensively used by Bennett and Rice, among others. We mention some results of this approach for the sake of completeness.

Expressing the output of a nonlinear device in response to an input u as $f(u)$, it is possible to find the (possibly two-sided) Laplace transform of $f(u)$, denoted $F(s)$, or,

$$F(s) = \int_{-\infty}^{\infty} e^{-su} f(u)\, du. \tag{18}$$

The inverse transform is then

$$f(u) = \frac{1}{2\pi j} \int_C e^{us} F(s)\, ds \tag{19}$$

where C is some suitably chosen contour of integration. If $F(s)$ exists, then (19) is an explicit expression for $f(u)$ which may be used to advantage. In the case of solving for modulation products, $f(u)$, written explicitly in x and y, may thus be used directly to compute the double series coefficients.

To compute A_{mn}, for example, using double Fourier Series expansion in response to an input $u = P \cos x + Q \cos y$, one has

$$\begin{aligned} A_{mn} &= \frac{\epsilon_m \epsilon_n}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos mx \cos ny\, dx\, dy \left[\frac{1}{2\pi j} \int_C e^{s(P\cos x + Q\cos y)} F(s)\, ds \right]^\dagger \\ &= \frac{\epsilon_m \epsilon_n}{8\pi^3 j} \int_C F(s)\, ds \int_{-\pi}^{\pi} e^{sP\cos x} \cos mx\, dx \int_{-\pi}^{\pi} e^{sQ\cos y} \cos ny\, dy. \end{aligned}$$

† ϵ_m is the Neumann factor, defined as:

$$\begin{aligned} \epsilon_m &= 1 \qquad m = 0 \\ \epsilon_m &= 2 \qquad m = 1, 2, \cdots . \end{aligned}$$

Since

$$\frac{j^{-n}}{\pi}\int_0^\pi e^{jz\cos\alpha}\cos n\alpha\, d\alpha = J_n(z) \tag{20}$$

and letting $s = j\omega$, one obtains,

$$A_{mn} = \frac{\epsilon_m\epsilon_n}{2\pi} j^{m+n}\int_C F(j\omega)J_m(P\omega)J_n(Q\omega)\, d\omega \tag{21}$$

and the required coefficients are evaluated by contour integration.

The above result can readily be generalized. For example, where there is a dc bias of b units superimposed on the $P\cos x + Q\cos y$ in the input, (i.e., $u = b + P\cos x + Q\cos y$), the net result is to insert a factor $e^{jb\omega}$ under the integral of (21).

Inputs of the form $u = b + \sum_{i=1}^{r} P_i \cos x_i$ will result in coefficients of the form

$$A_{n_1,\dots n_r} = \frac{\prod_{i=1}^{r}\epsilon_{n_1}\, j\exp\sum_{i=1}^{r} n_i}{\pi}\int_C e^{jb\omega}\,F(j\omega)\prod_{i=1}^{r}J_{n_1}(P_i\omega)\, d\omega \tag{22}$$

whenever $f(u)$ is Laplace transformable. The contour C is a function only of the nonlinear device, as may be expected. The Laplace transform of several ordinarily encountered ideal nonlinear devices, as well as their associated contour of integration C has been given in Rice[11] in his appendix 4A.

The nonlinear devices expressed as in (19) may be used fruitfully in certain investigation in noise problems. These are briefly described here. Reference may be made to Rice's classic papers of 1944 and 1945.[11]

For inputs that include narrowband noise, the input waveform will be of form

$$u = R\cos(\omega_m t + \theta) \qquad R \geqq 0$$

where R and θ are functions of time whose variation is slow as compared to $\cos\omega_m t$. ($\omega_m/2\pi$ is approximately the midband frequency.)

The output $f(u)$ then is

$$f(u) = \frac{1}{2\pi}\int_C F(j\omega)\exp[j\omega R\cos(\omega_m t + \theta)\, d\omega].$$

By means of the relation

$$e^{jz\cos\varphi} = \sum_{n=0}^{\infty}\epsilon_n j^n\cos n\varphi J_n(z) \tag{23}$$

the equation above may be written:

$$f(u) = \sum_{n=0}^{\infty} A_n(R) \cos (n\omega_m t + n\theta) \tag{24}$$

where

$$A_n(R) = \epsilon_n \frac{j^n}{2\pi} \int_C F(j\omega) J_n(\omega R)\ d\omega. \tag{25}$$

In this representation, important conclusions may be reached concerning the properties of the output without undertaking laborious computations. For $A_n(R)$ whose variation is of the order of that of R, the output spectrum has bands which are centered at f_m, $2f_m \cdots$. A narrow-band filter centered about nf_m will then yield a slowly varying cosine wave with envelope $A_n(R)$. A narrow-band low-pass filter will yield the level $A_0(R)$.

In some cases the probability density function $P(R)$ of R is known. (For narrow-hand Gaussian noise, for example, $P(R)$ is the Rayleigh distribution.) The probability density of the output envelope $A_n(R)$ is simply:

$$P[A_n(R)] = \frac{P(R)}{\left| \dfrac{dA_n(R)}{dR} \right|}. \tag{26}$$

Another application in which the representation of (19) is useful is the calculation of the autocovariance function of the output of a zero-memory nonlinear device. From this the output power spectrum is found by taking the Fourier cosine transform.

The autocovariance function of the output is:

$$\Psi(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f[u(t)] f[u(t + \tau)]\ dt\,. \tag{27}$$

By (20):

$$\Psi(\tau) = \lim_{T\to\infty} \frac{1}{4\pi^2 T} \int_0^T \int_C F(j\omega_1) \exp [j\omega_1 u(t)]\ d\omega_1 \int_C F(j\omega_2) \cdot \exp [j\omega_2 u(t + \tau)]\ d\omega_2\ dt\,.$$

If an exchange of limits is justifiable, the above becomes

$$\Psi(\tau) = \frac{1}{4\pi^2} \int_C F(j\omega_1)\ d\omega_1 \int_C F(j\omega_2)\ d\omega_2 \cdot \left[\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp [j\omega_1 u(t) + j\omega_2 u(t + \tau)]\ dt \right].$$

The quantity in the bracket is the (time) average value of

$$\exp j[\omega_1 u(t) + \omega_2 u(t + \tau)]$$

which, in the event that $u(t)$ satisfies the ergodic hypothesis, is equal to the characteristic function of the two variables $u(t)$ and $u(t + \tau)$. Denoting this quantity by $g(\omega_1, \omega_2, \tau)$, one has:

$$\Psi(\tau) = \frac{1}{4\pi^2} \int_C F(j\omega_1)\, d\omega_1 \int_C F(j\omega_2) g(\omega_1, \omega_2, \tau)\, d\omega_2. \qquad (28)$$

This gives an interesting approach to the computation of the output autocovariance function.

It is interesting to note, incidentally, that the characteristic function of $u(t) = P \cos pt$ is

$$J_0\left(P\sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos p\tau}\right),$$

and for

$$u(t) = P \cos pt + Q \cos qt$$

where p and q are incommensurable, the characteristic function is

$$J_0\left(P\sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos p\tau}\right) \times J_0\left(Q\sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos q\tau}\right).$$

Here, as elsewhere, one is limited by his ability to integrate. The autocovariance function, however, has been solved for particular nonlinear characteristics, for example, the square-law device.

IV. NUMERICAL SOLUTIONS AND BENNETT FUNCTIONS

Since it has not yet been found possible to express the modulation coefficients in a more general case in terms of known functions, it is often necessary to resort to numerical computations. The numerical approaches have been tackled by Sternberg, Kaufman, Feuerstein, Shipman, among others.[13-19] Some of their results have been tabulated and a class of generalized functions encountered in these investigations are christened Bennett functions.[14]

The original approach of Sternberg and Kaufman is along the lines of direct integration, summarized below.

If the output $f(u)$ can be expressed in the form of a continuous $N + 1$-sided polygonal function over a closed interval $-a \leqq u \leqq a$, i.e.,

$$f(u) = f(-a) + \sum_{i=1}^{N} g_i U_{-2}(u - u_i) \qquad i = 1, \cdots N \qquad (29)$$

where u_i and g_i are constants, u_i being the "break-points" of the polygo-

nal function

$$-a \leqq u_1 < u_2 < u_3 \cdots u_N < a \tag{30}$$

and $U_{-2}(u - u_i)$ are unit ramp functions:

$$\begin{aligned} U_{-2}(u - u_i) &= 0 & u &< u_i \\ U_{-2}(u - u_i) &= u - u_i & u &\geqq u_i \qquad (i = 1, 2 \cdots N). \end{aligned} \tag{31}$$

If the input is of form $u = P \cos x + Q \cos y$, one can confine his attention piecewise to N functions of the type

$$f_i(x,y) = f(P \cos x + Q \cos y;\, u_i) \qquad i = 1, 2 \cdots N. \tag{32}$$

The over-all function is then

$$f(x,y) = f(-a) + \sum_{i=1}^{N} g_i f_i(x,y). \tag{33}$$

Factoring out P in each term, and introducing parameters $h_i = u_i/P$, $k = Q/P$, we express $f_i(x,y)$ as the double Fourier series:

$$f_i(x,y) = \tfrac{1}{2} P A_{00}(h_i,k) + P \sum_{m,n=0}^{\infty}{}^{*} A_{\pm mn}(h_i,k) \cos (mx \pm ny)\dagger \tag{34}$$

where

$$A_{\pm mn}(h_i,k) = \frac{1}{2\pi^2 P} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_i(x,y) \cos (mx \pm ny)\, dx\, dy,$$

$$m,n = 0, 1, 2 \cdots;\ i = 1, 2 \cdots N$$

and for $f(x,y)$ we carry out another expansion:

$$f(x,y) = \tfrac{1}{2} C_{00} + \sum_{m,n=0}^{\infty}{}^{*} C_{\pm mn} \cos (mx \pm ny). \tag{35}$$

The C's and the A's are then related by

$$\begin{aligned} \tfrac{1}{2} C_{00} &= f(-a) + \tfrac{1}{2} P \sum_{i=1}^{N} g_i A_{00}(h_i,k) \\ C_{\pm mn} &= P \sum_{i=1}^{N} g_i A_{\pm mn}(h_i, k). \end{aligned} \tag{36}$$

As $A_{+mn}(h_i,k) = A_{-mn}(h_i,k)$ for all m and n, the $\pm$ sign can be dropped.

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† $\sum_{m,n=0}^{\infty}{}^{*}$ denotes a summation without the A_{00} term; in addition, terms whose index is such that $m \cdot n = 0$ are to be weighed by a factor of $\frac{1}{2}$.

By considering the function

$$f_i(x,y) = \begin{cases} P(\cos x + k\cos y - h_i); & \cos x + k \cos y \geqq h_i \\ 0 & \cos x + k\cos y < h_i \\ & i = 1, 2 \cdots N \end{cases}$$

the zones over which integration for the evaluation of A_{mn} must be carried out is seen to be bounded by the curve

$$\cos x + k\cos y = h; \qquad h = h_i$$

over the closed square

$$R_0 : \begin{cases} -\pi \leqq x \leqq \pi \\ -\pi \leqq y \leqq \pi. \end{cases}$$

Five cases need to be considered; two are degenerate:

$$\text{(d1)} \quad 1 + k \leqq h$$

$$\text{(d2)} \quad -(k+1) \geqq h$$

In the first instance the integrand vanishes everywhere except possibly over a set of zero measure, and hence the coefficients are identically zero. In the second instance the integration is to be carried out throughout the zone (excepting possibly a set of zero measure), which means the output is the same as the input except for a constant multiplying factor.

The three nondegenerate cases are:

$$(i) \quad h < 1 + k, \quad \text{or}$$
$$h > 1 - k.$$

The integral here is to be carried out over a zone R of the x,y plane bounded by a closed curve lying wholly within R_0.

$$(ii) \quad h \geqq k - 1, \quad \text{or}$$
$$h \leqq 1 - k$$

The integral here is to be carried out over a zone R bounded by two open curves (i.e., two opposite segments of the boundary of R also constitute the boundary of R_0).

$$(iii) \quad h < k - 1, \quad \text{or}$$
$$h > -(k+1)$$

The area of integration is bounded by four open curves. The integra-

tion can thus be carried out for these nondegenerate cases by formulas of the type:

$$A_{mn}(h,k) = \frac{2}{\pi^2} \int_{R_1} \left[\int (\cos x + k \cos y - h) \cos mx \, dx \right] \cos ny \, dy + \frac{2}{\pi^2} \int_{R_2} \left[\int (\cos x + k \cos y - h) \cos mx \, dx \right] \cos ny \, dy \qquad m, n = 0, 1, 2 \cdots$$

$$A_{mn}(h,k) = \frac{2}{\pi^2} \int_{R_3} \left[\int (\cos x + k \cos y - h) \cos ny \, dy \right] \cos mx \, dx + \frac{2}{\pi^2} \int_{R_4} \left[\int (\cos x + k \cos y - h) \cos ny \, dy \right] \cos mx \, dx \qquad m, n = 0, 1, 2 \cdots$$

where R_1, R_2, R_3, R_4 are zones appropriate for each of the cases.

It is seen that the inner integrals can be performed, after which suitable manipulation will yield a set of recurrence relationship first derived by Rice.[20] Except for misprints, they are:

$$\begin{aligned}
(m - n + 3) A_{m+1,n-1} &\equiv -(m + n - 3) A_{m-1,n-1} + 2mhA_{m,n-1} \\
&\quad - 2mkA_{mn} \qquad m,n \geqq 1 \\
(m + n + 1) A_{mn} &\equiv -(m - n - 3) A_{m-2,n} - 2(m - 1)kA_{m-1,n-1} \\
&\quad + 2(m - 1)hA_{m-1,n} \qquad m \geqq 2, n \geqq 1 \\
(n + m + 1) A_{mn} &\equiv -(n - m - 3) A_{m,n-2} \\
&\quad - 2(n - 1) \frac{1}{k} A_{m-1,n-1} + 2(n - 1) \frac{h}{k} A_{m,n-1} \\
&\quad m \geqq 1, n \geqq 2 \\
(n - m + 3) A_{m-1,n+1} &\equiv -(n + m - 3) A_{m-1,n-1} \\
&\quad + 2n \frac{h}{k} A_{m-1,n} - 2n \frac{1}{k} A_{mn} \qquad m,n \geqq 1 .
\end{aligned} \tag{37}$$

With the aid of these relationships, the higher-order coefficients can be expressed in terms of the first four coefficients $A_{00}(h,k)$, $A_{10}(h,k)$, $A_{01}(h,k)$ and $A_{11}(h,k)$.

For cases such as the ideal limiter, the antisymmetric condition

$f(-u) = -f(u)$ is observed; u_i are now symmetric and the "gains" g_i are antisymmetric. Here:

$$\begin{aligned} A_{00}(-h,k) &\equiv A_{00}(h,k) + 2h \\ A_{10}(-h,k) &\equiv 1 - A_{10}(h,k) \\ A_{01}(-h,k) &\equiv k - A_{01}(h,k) \\ A_{mn}(-h,k) &\equiv (-1)^{m+n} A_{mn}(h,k) \qquad (m+n>1). \end{aligned} \tag{38}$$

The function $A_{mn}(h,k)$ are called by Sternberg the Bennett functions of multiplicity two and order m,n. In part II of Sternberg's paper,[14] the functions $A_{00}(h,k)$, $A_{10}(h,k)$, $A_{01}(h,k)$ and $A_{11}(h,k)$, have been tabulated for h between -2 and $+2$ in 0.2 steps and k with values of 0.001, 0.01, 0.1 and 1.0. The values $A_{20}(h,k)$, $A_{02}(h,k)$, $A_{30}(h,k)$, $A_{21}(h,k)$, $A_{12}(h,k)$, $A_{03}(h,k)$ are tabulated for k of 0.1 and 1.0. All values are tabulated to six decimal places. The accuracy of the first set of tables is held to be to one unit in the last place, while for the second set the accuracy is about three units in the last place.

The above approach is extendable to devices with continuous and smooth characteristics if it can be approximated in a piece-wise linear form. As long as the characteristic may be approximated to within a pre-chosen $\epsilon > 0$ uniformly on the interval $-a \leqq u \leqq a$ by

$$S(u,\epsilon) = f(-a) + \sum_{i=1}^{N(\epsilon)} g_i U_{-2}(u - u_i), \tag{39}$$

Sternberg and Kaufman show that the approximate modulation product amplitudes computed as per (33), (34) and (35) will not differ from the true values by more than $4\epsilon/\pi$ in all cases, and the output will be within ϵ of the true value for all time if it is obtained by summing over the approximate expansions.

For a symmetrical ideal limiter

$$f(u) = \begin{cases} -gu_0 & u \leqq -u_0 \\ gu & -u_0 < u \leqq u_0 \\ gu_0 & u \geqq u_0 \end{cases} \qquad 0 < u_0 < 2P,\ g > 0.$$

The approaches described above can be applied to the range

$$-a \leqq x \leqq a$$

$$a \geqq 2P.$$

Sternberg, in part II of his paper,[14] gives the results relating the coefficients C_{mn} and A_{mn} as:

$$C_{\pm mn} = 0 \qquad m + n = 0, 2, 4 \cdots \tag{40a}$$

$$C_{10} = Pg[1 - 2A_{10}(h,k)] \tag{40b}$$

$$C_{01} = Pg[k - 2A_{01}(h,k)] \tag{40c}$$

$$C_{\pm mn} = -2PgA_{mn}(h,k) \qquad m + n = 3, 5, 7, \cdots. \tag{40d}$$

Here $H = u_0/P$, $k = Q/P$ and input are in the form

$$u(t) = P \cos (pt + \theta p) + Q \cos (qt + \theta q)$$
$$0 < P < P + Q \leqq 2P.$$

In Ref. 17, Bennett functions of the νth kind, denoted $A_{mn}^{(\nu)}(k)$, are defined. These are the coefficients for the output of a νth law rectifying function in response to a two-frequency input. ν is usually taken to be an integer. $A_{mn}^{(\nu)}(k)$ for $\nu = 1, 2$ have been tabulated.[17] Bennett functions of a given kind can be obtained from those of the lower kinds by means of recursion formulas.

By extending the above, Bennett functions with multiplicities of three or higher (i.e., modulation coefficients when the input has three or more distinct frequency components) can readily be defined. For input of the form

$$u(t) = P(\cos x + k_1 \cos y + k_2 \cos z) \tag{41}$$

for example, the output from a piece-wise linear nonlinear element can be expressed in terms of the Bennett functions of the first kind,

$$A_{mnl}(h, k_1, k_2), \qquad h = \frac{u_0}{P}$$

where, as before, u_0 is the breakpoint for an individual segment. Similarly, for a νth law rectifier subjected to inputs of the form (41), Bennett functions of the νth kind

$$A_{mnl}^{(\nu)}(h, k_1, k_2)$$

can be defined.

A number of interesting relations have been derived for the three frequency Bennett functions of the νth kind.[19] These include recurrence relations and integrals linking three-frequency Bennett functions with two-frequency ones. No tabulation of the three or more frequency Bennett functions is known to have been attempted.

Relationships between the "Fourier" representation and the "Laplace" representations for nonlinear characteristics have also been revealed by Feuerstein.[18] He has shown that, in many cases, the contour integration in the "Laplace" formulation can be reduced to integrals

over the infinite real line. The results are generalized Weber-Schafheitlin integrals of the form (16) and (17). This is perhaps not a surprising result from intuitive grounds.

It is interesting to note, however, that for a ν-law rectifier, the Bennett function of the νth kind of arbitrary multiplicity is given by

$$A_{m_0}{}^{(\nu)}{}_{,\cdots m_N}(h,k_1, \cdots, k_N) = \nu! j^{M-\nu-1} \frac{2}{\pi} \int_0^\infty \lambda^{-(\nu+1)} \cos \lambda h \cdot \prod_{i=0}^{N} J_{m_i}(k_i\lambda)\, d\lambda \tag{42a}$$

for ν integer,
$M \geqq \nu + 1$, and
$M + \nu$ odd,

and

$$A_{m_0}{}^{(\nu)}{}_{,\cdots m_N}(h,k_1, \cdots, k_N) = \nu! j^{M-\nu-2} \frac{2}{\pi} \int_0^\infty \lambda^{-(\nu+1)} \cdot \sin \lambda h \prod_{i=0}^{N} J_{m_i}(k_i\lambda)\, d\lambda \tag{42b}$$

for ν integer,
$M \geqq \nu$, and
$M + \nu$ even,

where $M = \sum_{i=0}^{N} m_i$ and $k_0 = 1 \geqq k_i$.

By these formulas, the generalized integrals of (17) are related directly to Bennett functions.

Feuerstein in fact did not stop with the considerations of integer ν. The formulas of the Bennett functions of the νth kind, with noninteger ν and with M taking in values other than those shown above, are related to the generalized integrals of (17), though in a more complicated form.

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